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## ABSTRACT

This article describes eight formulations for the *Generalized Minimum Spanning Tree Problem* (GMSTP). Relationships between the polytopes of their linear relaxations are established. It is shown that four of these polytopes are strictly included in the remaining ones. This analysis suggests which formulations should be preferred for the construction of a branch-and-cut algorithm and for the evaluation of heuristics.



**Keywords** : spanning trees, integer linear programming, polytopes.



## RÉSUMÉ

Cet article décrit huit formulations pour le problème de l'arbre de recouvrement généralisé. On établit des relations entre les polytopes des relaxations linéaires de ces formulations. On démontre que quatre de ces polytopes sont strictement inclus dans les quatre autres. Cette analyse permet de déterminer les meilleures formulations pour la construction d'un algorithme de branch-and-bound et pour l'évaluation d'heuristiques.



**Mots clefs** : arbres de recouvrement, programmation linéaire en nombres entiers, polytopes.

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# 1 Introduction

The purpose of this article is to analyze and compare eight integer linear programming formulations for the *Generalized Minimum Spanning Tree Problem* (GMSTP) defined as follows. Let $G = (V, E)$ be an undirected graph where $V = \{1, \ldots, n\}$ is a vertex set partitioned into *clusters* $V_k, k \in K$ and $E = \{\{i, j\} : i \in V_k, j \in V_l, k, l \in K, k \neq l\}$. With each edge $\{i, j\} \in E$ is associated a cost $c_{ij}$. The GMSTP consists of determining a minimum cost tree including *exactly* one vertex from each cluster. Applications of the problem are encountered in telecommunications where metropolitan and regional area networks must be interconnected by a tree containing a gateway from each cluster (Myung, Lee and Tcha [3]). These authors have proved that the GMSTP is $NP$-hard and have provided four formulations. We introduce four additional formulations and we provide a comparative analysis of the polytopes associated with the linear relaxations of all eight formulations. We show that the linear relaxations of four of these formulations

are equivalent in the sense that their associated polytopes coincide and they thus yield the same optimal value. These four formulations also dominate all others with respect to polytope inclusion. Since the best four formulations provide the strongest known lower bounds on the optimal GMSTP value, they emerge as natural candidates for the construction of an exact algorithm and for the evaluation of heuristics.

The remainder of this article is organized as follows. In Section 2 we introduce some notation common to all formulations. The eight models are grouped under four headings. Section 3 contains two undirected formulations based on tree properties and already introduced by Myung et al. The corresponding directed formulations, one contained in Myung et al. and a new one, are described in Section 4. A flow formulation due to Myung et al. and its projection in a restricted space are given in Section 5. Finally, we provide in Section 6 a strengthening of an undirected formulation given by Myung et al. and its directed counterpart. A conclusion follows in Section 7.

## 2 Notations

Given an undirected graph $G(V, E)$ , define $\delta(S) = \{\{i, j\} \in E : i \in S, j \notin S\}$ and $E(S) = \{\{i, j\} \in E : i \in S, j \in S\}$. It is often useful to consider the directed graph $G' = (V, A)$ associated with $G = (V, E)$, where $A$ is an arc set containing two arcs $(i, j)$ and $(j, i)$ for each edge $\{i, j\}$ of $E$. We also define $\delta^+(S) = \{(i, j) \in A : i \in S, j \notin S\}$, $\delta^-(S) = \{(i, j) \in A : i \notin S, j \in S\}$ and $A(S) = \{(i, j) \in A : i \in S, j \in S\}$. For simplicity we write $\delta^-(i)$ and $\delta^+(i)$ instead of $\delta^-(\{i\})$ and $\delta^+(\{i\})$.

We define the following binary variables:

$$x_{ij} = \begin{cases} 1 & \text{if edge } \{i,j\} \in E \text{ belongs to the solution} \\ 0 & \text{otherwise,} \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if vertex } i \in V \text{ belongs to the solution} \\ 0 & \text{otherwise,} \end{cases}$$

$$w_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \in A \text{ belongs to the solution} \\ 0 & \text{otherwise.} \end{cases}$$

We use the vector notation $x = (x_{ij})$, $y = (y_i)$ and $w = (w_{ij})$ and the notations $x(E') = \sum_{\{i,j\} \in E'} x_{ij}$ for $E' \subseteq E$, $y(V') = \sum_{i \in V'} y_i$ for $V' \subseteq V$, and $w(A') = \sum_{(i,j) \in A'} w_{ij}$ for $A' \subseteq A$.

We denote by $\mathcal{P}_f$ the polytope associated with the linear relaxation of a mixed or pure binary linear formulation $f$ projected in the space of $x$ and $y$ variables. Formally, this linear relaxation is obtained by replacing all binary conditions by constraints requiring the corresponding variables to take values in $[0,1]$. Then $\mathcal{P}_f$ is the set of all feasible solutions to the linear relaxation. Since we are only interested in comparing polytopes associated with various models, we omit the objective "minimize $\sum_{\{i,j\} \in E} c_{ij} x_{ij}$" from all subsequent formulations. Also, the binary conditions on the $x, y$ and $w$ variables are implicitly assumed in all models.

# 3 Two formulations based on tree properties

Myung et al. have provided two basic formulations for the GMSTP. In the first, called *ucut*, connectivity is ensured by *cutset constraints* of the form $x(\delta(S)) \geq y_i + y_j - 1$ $(i \in S \subset V, j \notin S)$, whereas in the second, called *usub*, cycles are prevented through *subpacking constraints* of the form $x(E(S)) \leq y(S \backslash \{i\})$ $(i \in S, S \subset V, 2 \leq |S| \leq n-1)$. The undirected cutset formulation uses the fact that a feasible *Generalized Spanning Tree* (GST) is a connected subgraph of $G$ containing one vertex per cluster and $|K| - 1$ edges:

***Undirected cutset formulation (ucut)***

$$y(V_k) = 1 \qquad (k \in K), \qquad (1)$$

$$x(E) = |K| - 1, \qquad (2)$$

$$x(\delta(S)) \geq y_i + y_j - 1 \qquad (i \in S \subset V, j \notin S). \qquad (3)$$

A GST can also be defined as an acyclic subgraph of $G$ containing one vertex per cluster and $|K| - 1$ edges:

***Undirected subpacking formulation (usub)***

$$y(V_k) = 1 \qquad (k \in K), \qquad (1)$$

$$x(E) = |K| - 1, \qquad (2)$$

$$x(E(S)) \leq y(S \backslash \{i\}) \qquad (i \in S, S \subset V, 2 \leq |S| \leq n-1). \qquad (4)$$

Myung et al. have already proved that $\mathcal{P}_{usub} \subseteq \mathcal{P}_{ucut}$. The example depicted in Figure 1, provided by Magnanti and Wolsey [2] in the context of the *Minimum Spanning Tree Problem*, can also be used to show that this inclusion is strict. (In all figures, the positive $x_{ij}$ values are shown on the edges or arcs and the positive $y_i$ values are shown on the vertices). Indeed (1), (2) and (3) are satisfied while (4) is violated for $S = \{3, 4, 5\}$.

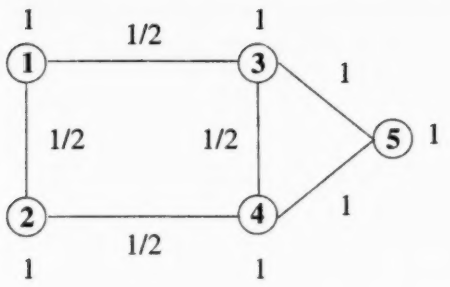


Figure 1: Example showing that $\mathcal{P}_{usub} \subset \mathcal{P}_{ucut}$

# 4 Two directed formulations based on arborescence properties

It is always possible to transform the undirected GMSTP into a *Generalized Minimum Spanning Arborescence Problem* (GMSAP) which consists of determining a minimum cost arborescence in a directed graph $G = (V, A)$, rooted in a given cluster $V_1$, and containing exactly one vertex per cluster. The transformation is achieved by replacing each edge $\{i, j\}$ by the opposite arcs $(i, j)$ and $(j, i)$ each having a cost $c_{ij}$. We now propose the following formulation:

***Directed cutset formulation (dcut)***

$$y(V_k) = 1 \qquad (k \in K), \qquad (1)$$

$$x(E) = |K| - 1, \qquad (2)$$

$$w(\delta^-(S)) \geq y_i \qquad (i \in S \subseteq V \backslash V_1), \qquad (5)$$

$$w_{ij} \leq y_i \qquad (i \in V_1, \quad j \notin V_1), \qquad (6)$$

$$w_{ij} + w_{ji} = x_{ij} \qquad (\{i, j\} \in E). \qquad (7)$$

Formulation *dcut* can be viewed as a more compact rewriting of the $P_1^d$ formulation of Myung et al. (also denoted by *dcut'*) in which the constraints

$$w(\delta^-(V_1)) = 0 \tag{8}$$

and

$$w(\delta^-(V_k)) \leq 1 \quad (k \in K\backslash\{1\}) \tag{9}$$

are used instead of (2). To show the equivalence of *dcut* and $P_1^d$ we first prove the following lemma.

## Lemma 1.

a) Constraints (1), (2), (5), (7) imply

$$w(\delta^-(i)) = 0 \qquad (i \in V_1) \tag{10}$$

and

$$w(\delta^-(i)) = y_i \quad (i \in V\backslash V_1). \tag{11}$$

b) Constraints (1), (5), (7), (8), (9) imply (2).

## Proof.

a) First observe that constraint (5) for $S = \{i\}$ $(i \in V\backslash V_1)$ is $w(\delta^-(i)) \geq y_i$. To show that the equality must hold, suppose that $w(\delta^-(j)) > y_j$ for some $j \in V\backslash V_1$. Since (7) implies $x(E) = w(A) \geq \sum_{i \in V\backslash V_1} w(\delta^-(i))$, this would result in $w(A) > \sum_{i \in V\backslash V_1} y_i = |K| - 1$ by constraints (1), thus contradicting (2). Now

$$\begin{aligned} x(E) = w(A) &= \sum_{j \in V\backslash V_1} w(\delta^-(j)) + w(\delta^-(V_1)) \\ &= \sum_{j \in V\backslash V_1} y_j + w(\delta^-(V_1)) \quad \text{by (11)}, \\ &= |K| - 1 + w(\delta^-(V_1)) \quad \text{by (1)}. \end{aligned}$$

Since (2) is satisfied it follows that $w(\delta^-(V_1)) = 0$ and (10) must also hold since there are no intracluster arcs.

b) Constraints (1), (5) and (9) imply $w(\delta^-(V_k)) = 1$ for $k \in K\backslash\{1\}$. Using (7) and (8), it follows that $x(E) = w(A) = \sum_{k \in K} w(\delta^-(V_k)) = |K| - 1$, and therefore (2) is satisfied. ∎

**Proposition 2.** $\mathcal{P}_{dcut} = \mathcal{P}_{dcut'}$.

**Proof.** Follows directly from Lemma 1.

Myung et al. state that $\mathcal{P}_{dcut} \subseteq \mathcal{P}_{usub}$, but this result is wrong. Indeed, Figure 2 depicts a solution belonging to $\mathcal{P}_{dcut} \backslash \mathcal{P}_{usub}$ since it satisfies (1), (2), (5) – (7) but violates (4) for $S = \{1, 3, 4, 6, 7, 8\}$ and $i = 8$.

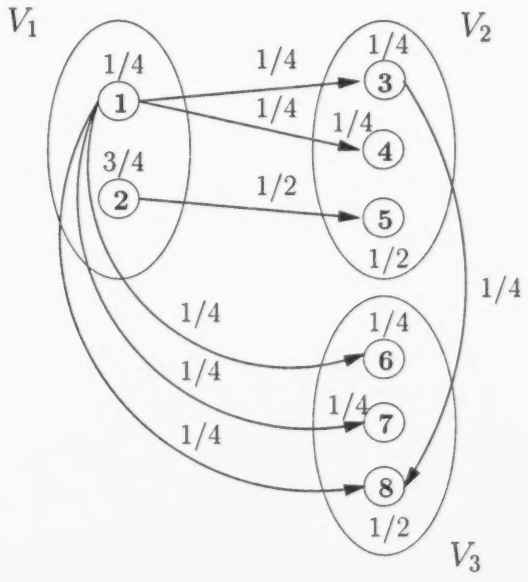


Figure 2: Example showing that $\mathcal{P}_{dcut} \backslash \mathcal{P}_{usub} \neq \emptyset$

Finally, the solution represented by Figure 3 clearly shows that $\mathcal{P}_{usub} \backslash \mathcal{P}_{dcut} \neq \emptyset$ since it satisfies constraints (1), (2) and (4) but violates (5) for $S = \{3\}$.

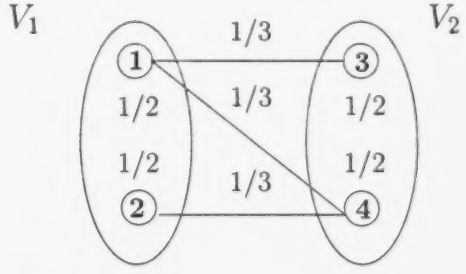


Figure 3: Example showing that $\mathcal{P}_{usub} \backslash \mathcal{P}_{dcut} \neq \emptyset$

We now propose a directed subpacking formulation derived from *usub*.

***Directed subpacking formulation (dsub)***

$$
\begin{aligned}
y(V_k) &= 1 && (k \in K), && (1)\\
x(E) &= |K| - 1, && && (2)\\
x(E(S)) &\leq y(S\backslash\{i\}) && (i \in S, S \subset V, 2 \leq |S| \leq n-1), && (4)\\
w_{ij} + w_{ji} &= x_{ij} && (\{i,j\} \in E), && (7)\\
w(\delta^-(j)) &= y_j && (j \in V\backslash V_1). && (11)
\end{aligned}
$$

We will show that $\mathcal{P}_{dsub} = \mathcal{P}_{dcut} \cap \mathcal{P}_{usub}$. To this end, we first prove the following lemma.

## Lemma 3.

a) Constraints (4), (7) imply (6).

b) Constraints (1), (2), (7) and (11) imply (10).

c) Constraints (1), (2), (7) and (11) imply

$$x(E(S)) + w(\delta^-(S)) = y(S\backslash V_1) \qquad (S \subseteq V). \qquad (12)$$

## Proof.

a) Constraints (6) follow directly from the application of (4) and (7) with $S = \{i,j\}$ $(i \in V_1, j \notin V_1)$.

b) Combining (1), (7) and (11) yields $x(E) = |K| - 1 + \sum_{i \in V_1} w(\delta^-(i))$. Constraint (2) successively implies $\sum_{i \in V_1} w(\delta^-(i)) = 0$ and (10).

c) For any $S \subseteq V$, the arc set $A$ can be partitioned into $\{A(S), \delta^-(S), \{\delta^-(i)) : i \notin S\}\}$, which means that $w(A)$ can be written as

$$w(A) = x(E) = x(E(S)) + w(\delta^-(S)) + \sum_{i \notin S} w(\delta^-(i)) = |K| - 1.$$

Since (10) and (11) hold, it follows that

$$x(E(S)) + w(\delta^-(S)) = |K| - 1 - y(V\backslash(S \cup V_1)) = |K| - 1 - y(V\backslash V_1) + y(S\backslash V_1).$$

Constraints (12) follow since summing up (1) over $k \in K\backslash\{1\}$ yields $y(V\backslash V_1) = |K| - 1$. ∎

**Proposition 4.** $\mathcal{P}_{dsub} = \mathcal{P}_{dcut} \cap \mathcal{P}_{usub}$.

**Proof.**

a) $\mathcal{P}_{dsub} \subseteq \mathcal{P}_{dcut}$. Let $(x, y) \in \mathcal{P}_{dsub}$. We must show that there exists a vector $w$ such that (5) and (6) are satisfied. The validity of constraint (6) follows from Lemma 3a. Now, constraints (12) implied by Lemma 3c mean that (4) can be rewritten as $y(S\backslash V_1) - w(\delta^-(S)) \leq y(S\backslash\{i\})$ with $i \in S \subseteq V\backslash V_1$, which is equivalent to (5).

b) $\mathcal{P}_{dsub} \subseteq \mathcal{P}_{usub}$. Obvious.

c) $\mathcal{P}_{dcut} \cap \mathcal{P}_{usub} \subseteq \mathcal{P}_{dsub}$.

Lemma 1 means that if $(x, y) \in \mathcal{P}_{dcut} \cap \mathcal{P}_{usub}$ then (11) are satisfied. ∎

# 5 Two flow based formulations

The last directed formulation proposed by Myung et al. uses integer variables $f_{ij}^k$ equal to the amount of flow on arc $(i, j)$ traveling from the root cluster $V_1$ to every cluster $V_k$ ($k \in K\backslash\{1\}$). Thus every vertex $i$ of $V_1$ is a potential source with supply $y_i$, and every vertex $j$ of $V\backslash V_1$ is a sink with demand $y_j$.

***Multiflow formulation (mflow)***

$$
\begin{aligned}
y(V_k) &= 1 && (k \in K), \quad (1)\\
w_{ij} + w_{ji} &= x_{ij} && (\{i,j\} \in E), \quad (7)\\
w(\delta^-(V_1)) &= 0, && (8)\\
w(\delta^-(V_k)) &\leq 1 && (k \in K\backslash\{1\}), \quad (9)\\
f^k(\delta^+(i)) - f^k(\delta^-(i)) &= \left\{ \begin{array}{ll} y_i & (i \in V_1),\\ -y_i & (i \in V_k),\\ 0 & (i \notin V_1 \cup V_k), \end{array} \right\} && (k \in K\backslash\{1\}), \quad (13)\\
0 \leq f^k_{ij} &\leq w_{ij} && ((i,j) \in A, k \in K\backslash\{1\}), \quad (14)
\end{aligned}
$$

where $f^k(E') = \sum_{\{i,j\} \in E'} f^k_{ij}$. To derive further relationships between polytopes, namely between $\mathcal{P}_{mflow}$ and $\mathcal{P}_{dcut}$, it is useful to view $mflow$ as $|K| - 1$ independent flow problems in which vertices $i$ of $V_1$ have nonnegative supplies $y_i$, vertices $i$ of $V_k$ $(k \in K\backslash\{1\})$ have nonnegative demands $y_i$ and the remaining transshipment vertices have zero demands. From constraints (1), both the total supply and the total demand are equal to 1 and the Gale [1] necessary and sufficient feasibility condition for the existence of a feasible flow applies directly, namely,

$$
w(\delta^-(S)) \geq y(V_k \cap S) - y(V_1 \cap S) \qquad (S \subset V, 1 \leq |S| \leq n-1, \quad (15)
$$
$$
k \in K\backslash\{1\}).
$$

This yields the following *flow cut* formulation:

***Flow cut formulation (fcut)***

$$
\begin{aligned}
y(V_k) &= 1 && (k \in K), && (1)\\
w_{ij} + w_{ji} &= x_{ij} && (\{i,j\} \in E), && (7)\\
w(\delta^-(V_1)) &= 0, && && (8)\\
w(\delta^-(V_k)) &\leq 1 && (k \in K\backslash\{1\}), && (9)\\
w(\delta^-(S)) &\geq y(V_k \cap S) - y(V_1 \cap S) && (S \subset V, 1 \leq |S| \leq n-1, && (15)\\
& && \quad k \in K\backslash\{1\}). &&
\end{aligned}
$$

In Section 6, we use $fcut$ to show that $\mathcal{P}_{mflow} \subset \mathcal{P}_{dcut}$.

# 6 Two cluster subpacking formulations

A strengthening of *usub* is obtained by replacing

$$x(E(S)) \leq y(S\backslash\{i\}) \qquad (i \in S, S \subset V, 2 \leq |S| \leq n-1) \qquad (4)$$

with the *cluster subpacking* constraints

$$
\begin{aligned}
x(E(S)) \leq y(S\backslash V_k) \qquad & (S \subset V, 2 \leq |S| \leq n-1, && (16)\\
& \quad k \in K\backslash\{1\}). &&
\end{aligned}
$$

Note that (16) is dominated by

$$x(E(S')) \leq y(S'\backslash V_k) = y(S') - 1, \qquad (17)$$

where $S' = S \cup V_k$, since $x(E(S)) \leq x(E(S \cup V_k)) \leq y((S \cup V_k)\backslash V_k) = y(S\backslash V_k)$. In other words, any constraint (16) for which $S$ does not include a cluster is dominated by (17) where $S'$ includes at least one cluster.

We therefore arrive at the:

***Undirected cluster subpacking formulation (ucsub)***

$$
\begin{aligned}
y(V_k) &= 1 && (k \in K), \quad (1)\\
x(E) &= |K| - 1, && (2)\\
x(E(S)) &\leq y(S) - 1 && (S \subset V, 2 \leq |S| \leq n-1, \quad (17)\\
& && S \text{ includes at least one cluster } V_k).
\end{aligned}
$$

The directed counterpart of *ucsub* is given by

***Directed cluster subpacking formulation (dcsub)***

$$
\begin{aligned}
y(V_k) &= 1 && (k \in K), \quad (1)\\
x(E) &= |K| - 1 && (2)\\
w_{ij} + w_{ji} &= x_{ij} && (\{i,j\} \in E), \quad (7)\\
w(\delta^-(j)) &= y_j && (j \in V \backslash V_1), \quad (11)\\
x(E(S)) &\leq y(S) - 1 && (S \subset V, 2 \leq |S| \leq n-1, \quad (17)\\
& && S \text{ includes at least one cluster } V_k).
\end{aligned}
$$

It is clear that $\mathcal{P}_{dcsub} \subseteq \mathcal{P}_{ucsub}$. We will show that $\mathcal{P}_{dcsub} = \mathcal{P}_{ucsub}$. This result is interesting in the sense that contrary to what happens in some other network optimization problems (e.g., the Minimum Spanning Tree Problem [2]), the directed formulation is not stronger than the undirected formulation.

**Proposition 5.** $\mathcal{P}_{ucsub} = \mathcal{P}_{dcsub}$.

**Proof.** It is clear that $\mathcal{P}_{dcsub} \subseteq \mathcal{P}_{ucsub}$. To show the reverse inclusion, we prove as in Magnanti and Wolsey [2] that if $(x, y)$ satisfies (1), (2) and (17), then there exists a

nonnegative vector $w$ satisfying (7), (10) and (11). The system given by (7), (10) and (11) describes a flow feasibility problem in a directed graph $H$ defined as follows. The vertex set of $H$ is $V \cup E$ and arcs of infinite capacity are defined from each vertex $i$ to each edge $\{i, j\}$ incident to $i$. Vertices of $V_1$ have zero supply, vertices $i$ of $V \backslash V_1$ have a supply $y_i$ and vertices $\{i, j\}$ of $E$ have a demand $x_{ij}$. The Gale [1] necessary and sufficient conditions for the existence of a feasible flow $w$ can be expressed as

$$x(E(S)) \leq y(S \backslash V_1) \qquad (S \subset V, 2 \leq |S| \leq n-1), \qquad \textbf{(18)}$$

which is a special case of (17). The flow on arc $(i, \{i, j\})$ is equal to $w_{ji}$ (see Figure 4).

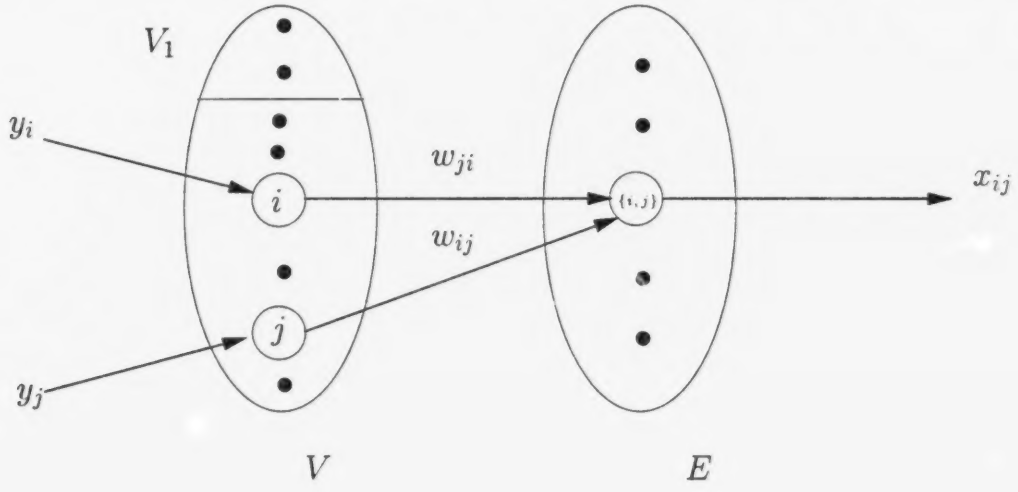


Figure 4: Graph $H$.

We now show that $\mathcal{P}_{dcsub} = \mathcal{P}_{fcut}$.

**Proposition 6.** $\mathcal{P}_{dcsub} = \mathcal{P}_{fcut}$

**Proof.**

a) Consider a point $(x, y) \in \mathcal{P}_{dcsub}$. By Lemma 3b, constraints (8) and (9) are satisfied. Using (12), constraints (17) can be shown to be equivalent to (15). Therefore $(x, y) \in \mathcal{P}_{fcut}$.

b) Consider a point $(x, y) \in \mathcal{P}_{fcut}$. We first show that (15) implies (5): if $S \subseteq V \backslash V_1$, then $S \cap V_1 = \emptyset$ and (15) can be rewritten as $w(\delta^-(S)) \geq y(S \cap V_k) \geq y_i$ if $i \in V_k$. Now, using Lemma 1b, (2) holds. The validity of (11) follows from Lemma 1a since (2) and (5) hold. Therefore $(x, y) \in \mathcal{P}_{dcsub}$ since (15) and (17) are equivalent. ∎

To complete the analysis, we show that $\mathcal{P}_{dcsub}$ (and thus $\mathcal{P}_{ucsub}$, $\mathcal{P}_{fcut}$ and $\mathcal{P}_{mflow}$) is strictly included in $\mathcal{P}_{dsub}$.

**Proposition 7.** $\mathcal{P}_{dcsub} \subset \mathcal{P}_{dsub}$.

**Proof.**

a) It is clear that $\mathcal{P}_{dcsub} \subseteq \mathcal{P}_{dsub}$ since constraints (17) are a subset of constraints (16) which themselves dominate constraints (4).

b) Finally, the solution depicted in Figure 5 shows that $\mathcal{P}_{dcsub} \neq \mathcal{P}_{dsub}$ since it satisfies constraints (1), (2), (4), (7) and (11) but violates (17) for $S = \{2, 3, 4, 5\}$. ∎

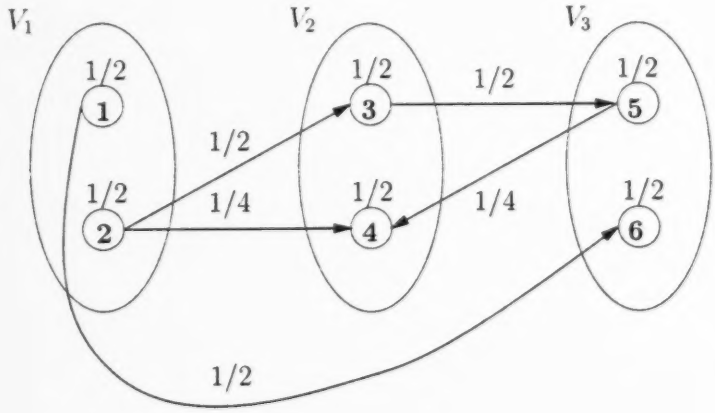


Figure 5: Example showing that $\mathcal{P}_{dcsub} \subset \mathcal{P}_{dsub}$.

# 7 Conclusion

We have provided eight formulations for the *Generalized Minimum Spanning Tree Problem*, four of which were known (Myung et al. [3]), and four new ones. We have shown that the $mflow$ formulation of Myung et al. and three of the new formulations yield the best linear relaxations. Among these, the *ucsub* formulation is the most compact in terms of the number of variables. The relationships between the eight formulations are depicted in Figure 6.

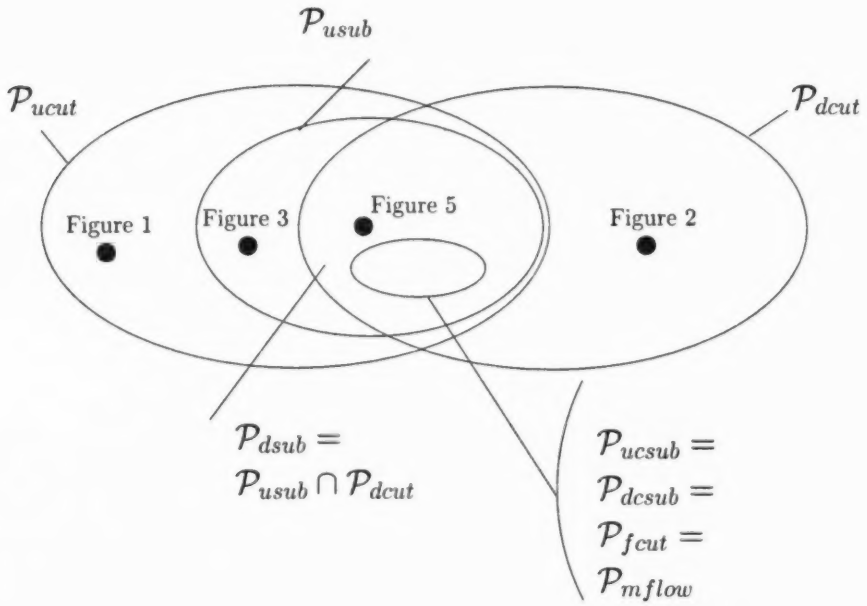


Figure 6: Relationships between the eight formulations.

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